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The C-matrix and the reality classification of the representations of the homogeneous Lorentz group: II. Decomposable representations of SO(3, 1)

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Abstract. General conditions are determined under which a decomposable representation of a group admits invertible C-matrices. Applying these results, the C-matrices admitted by the decomposable representations of the orthochronous proper Lorentz group SO(3, 1) are obtained and the representations are classified into three reality types.

1. Introduction

Continuing the theme of our previous paper (I) (Gopala Rao *et al* 1994), where we considered the reality classification of the irreps of the group SO(3, 1), we take up here the problem of classifying the decomposable (or completely reducible) representations of SO(3, 1) into the three reality types. The solution to this problem cannot be obtained by a trivial direct summation of the corresponding results of the irreducible case as the direct sum of irreps belonging to a given reality type need not necessarily be a representation of the same type.

In the following section, we obtain the general conditions under which a decomposable representation of an arbitrary group Γ admits invertible C-matrices. We also extend certain results of I concerning the potential reality of an irrep to the case of decomposable representations.

In section 3, we construct the invertible C-matrices associated with the decomposable representations of SO(3, 1) and classify them into the three reality types.

2. General considerations

Since we are interested in a discussion of the existence of a C-matrix associated with a decomposable representation $\mathbf{D}$ of a group Γ, we work in a basis in which $\mathbf{D}$ decouples as a direct sum of the irreps $\mathbf{D}_t$, $t = 1, 2, \ldots, n$. We also assume that the constituent irreps $\mathbf{D}_t$ of $\mathbf{D}$ have been rearranged such that $\mathbf{D} = \mathbf{D}_{\mathrm{III}} \oplus \mathbf{D}_{\mathrm{II}} \oplus \mathbf{D}_{\mathrm{I}}$, where $\mathbf{D}_{\mathrm{III}}$, $\mathbf{D}_{\mathrm{II}}$ and $\mathbf{D}_{\mathrm{I}}$ are, respectively, the direct sums of irreps of the third, second, and first kinds only. We partition the C-matrix associated with $\mathbf{D}$ into the block form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} & \cdots \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

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where a typical block matrix $\mathbf{C}_{tt'}$ has as many rows as the dimension of the irrep $\mathbf{D}_t$ and as many columns as the dimension of the irrep $\mathbf{D}_{t'}$. Then, on equating the corresponding sub-blocks in the defining relation $\mathbf{CD}(g) = \mathbf{D}^*(g)\mathbf{C}$, $g \in \Gamma$, of the C-matrices admitted by $\mathbf{D}$, we obtain the following equation:

$$\mathbf{C}_{tt'}\mathbf{D}_{t'}(g) = \mathbf{D}_t^*(g)\mathbf{C}_{tt'} \qquad (t, t' \text{ not dummies}) \tag{2.1}$$

which is valid for every $g \in \Gamma$. By the generalized Schur lemma (see, for example, Coleman, (1968)), any non-zero block $\mathbf{C}_{tt'}$ must necessarily be invertible. As a consequence of this result and the fact that two equivalent irreps must belong to the same reality type, it follows that $\mathbf{C}_{tt'} = 0$ if $\mathbf{D}_t$ and $\mathbf{D}_{t'}$ belong to different reality types. Therefore, the C-matrix admitted by $\mathbf{D} = \mathbf{D}_{\mathrm{III}} \oplus \mathbf{D}_{\mathrm{II}} \oplus \mathbf{D}_{\mathrm{I}}$ must necessarily be of the form $\mathbf{C} = \mathbf{C}_{\mathrm{III}} \oplus \mathbf{C}_{\mathrm{II}} \oplus \mathbf{C}_{\mathrm{I}}$, where the block-matrices $\mathbf{C}_{\mathrm{III}}$, $\mathbf{C}_{\mathrm{II}}$ and $\mathbf{C}_{\mathrm{I}}$ are the C-matrices admitted by the blocks $\mathbf{D}_{\mathrm{III}}$, $\mathbf{D}_{\mathrm{II}}$ and $\mathbf{D}_{\mathrm{I}}$, respectively. Second, a diagonal block $\mathbf{C}_{tt}$ can only be a zero-matrix when $\mathbf{D}_t$ is essentially complex, while an invertible block matrix $\mathbf{C}_{tt}$ exists when $\mathbf{D}_t$ is either potentially real or pseudo-real. On substituting for $\mathbf{D}_{t'}(g)$, from equation (2.1) in the equation $\mathbf{C}_{t't}\mathbf{D}_t(g) = \mathbf{D}_{t'}^*(g)\mathbf{C}_{t't}$, and applying the Schur lemma, we also note that, whenever $\mathbf{C}_{tt'}$ is invertible,

$$\mathbf{C}_{t't} = \alpha\mathbf{C}_{tt'}^{*-1} \tag{2.2}$$

where α is an arbitrary scalar.

As the blocks $\mathbf{C}_{\mathrm{I}}$ and $\mathbf{C}_{\mathrm{II}}$ can always be chosen so as to possess inverses (for example, by retaining only their diagonal blocks $\mathbf{C}_{tt}$), it is clear that the C-matrix is invertible if the block $\mathbf{C}_{\mathrm{III}}$ is invertible. A little reasoning leads to the following results (see the appendix for a proof).

Theorem 1. A decomposable representation $\mathbf{D}$ of a group Γ admits invertible C-matrices if, and only if, every essentially complex irrep $\mathbf{D}_i$ occurs in $\mathbf{D}$ a finite number of times and the multiplicities of $\mathbf{D}_i$ and $\mathbf{D}_i^*$ are equal.

In applying this theorem, i.e. in counting irreps, the multiplicity of an irrep $\mathbf{D}_i$ in $\mathbf{D}$ is to be taken as k if $\mathbf{D}_i$, and representations equivalent to $\mathbf{D}_i$, appear together k times in $\mathbf{D}$.

It is interesting to note the following results, which are in sharp contrast to the corresponding results of the irreducible case.

(i) Although every scalar multiple of a given C-matrix is also a C-matrix, all the C-matrices associated with a completely reducible representation are not necessarily scalar multiples of each other.

(ii) A C-matrix associated with a decomposable representation need not satisfy any of the two relations $\mathbf{CC}^* = \pm\mathbf{E}$.

We support the above remarks by an example. Consider the two-component representation $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_1$, where $\mathbf{D}_1$ is an irreducible pseudo-real representation of SO(3, 1). Then, (see I) the irrep $\mathbf{D}_1$ admits a real C-matrix satisfying $\mathbf{C}_1^2 = -\mathbf{E}$, and it is easily checked that

$$\mathbf{C} = \left(\begin{array}{c|c} \alpha_1\mathbf{C}_1 & \alpha_2\mathbf{C}_1 \\ \hline \alpha_3\mathbf{C}_1 & \alpha_4\mathbf{C}_1 \end{array}\right)$$

is a C-matrix associated with $\mathbf{D}$, where α_1, α_2, α_3 and α_4 are arbitrary real numbers. Note that this $\mathbf{C}$ satisfies $\mathbf{CC}^* = \mathbf{E}$ when $\alpha_1 = \alpha_4 = 0$, $\alpha_2 = -\alpha_3 = 1$, and $\mathbf{CC}^* = -\mathbf{E}$ when

$\alpha_1 = \alpha_4 = 1$, $\alpha_2 = \alpha_3 = 0$. However, with $\alpha_2 = \alpha_3 = 0$, $\alpha_1 \neq \alpha_4$, it satisfies neither $\mathbf{CC}^* = \mathbf{E}$ nor $\mathbf{CC}^* = -\mathbf{E}$. .

Similarly, we make the following observations on the problem of further sorting the completely reducible representations possessing C-matrices into first and second kinds.

(1) As in the case of irreps (I), the existence of the C-matrix which can be expressed in the form $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$ is the necessary and sufficient condition for the potential reality of a decomposable representation.

(2) *If a decomposable representation* $\mathbf{D}$ *is potentially real, there must necessarily exist* at least one C-matrix which satisfies $\mathbf{CC}^* = \mathbf{E}$.

(3) If none of the C-matrices admitted by a decomposable representation $\mathbf{D}$ satisfies the relation $\mathbf{CC}^* = \mathbf{E}$, then $\mathbf{D}$ must belong to the second kind.

(4) In expressing a C-matrix satisfying the relation $\mathbf{CC}^* = \mathbf{E}$ in the form $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$, we may note that all the results which are valid for an irrep are also valid for a decomposable representation.

(5) If two irreps $\mathbf{D}_t$ and $\mathbf{D}_{t'}$, of a group Γ satisfy equation (2.1) with an appropriate invertible matrix $\mathbf{C}_{tt'}$, then the direct sum representation $\mathbf{D}_t \oplus \mathbf{D}_{t'} = \mathbf{D}_t \oplus (\mathbf{C}_{tt'}^{-1}\mathbf{D}_t^*\mathbf{C}_{tt'})$ is equivalent to a real representation.

To see this, it is sufficient to note that with

$$\mathbf{T} = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} \mathbf{E} & -\mathrm{i}\mathbf{E} \\ \hline \mathbf{C}_{tt'}^{-1} & \mathrm{i}\mathbf{C}_{tt'}^{-1} \end{array}\right] \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} \mathbf{E} & \mathbf{C}_{tt'} \\ \hline \mathrm{i}\mathbf{E} & -\mathrm{i}\mathbf{C}_{tt'} \end{array}\right]$$

$\mathbf{T}^{-1}(\mathbf{D}_t \oplus \mathbf{D}_{t'})\mathbf{T}$ is real. (Note that this $\mathbf{T}$ is unitary whenever $\mathbf{C}_{tt'}$ is unitary). Therefore, it follows in particular (i.e. if $\mathbf{C}_{tt'} = \mathbf{E}$) that $\mathbf{D}_t \oplus \mathbf{D}_t^*$ is equivalent to a real representation for all representations $\mathbf{D}_t$. Combining this result with the fact that a representation $\mathbf{D}$ and its complex conjugate $\mathbf{D}^*$ must belong to the same reality type (it is easy to check this), we observe that when $\mathbf{D}$ and $\mathbf{D}^*$ are essentially complex or pseudo-real, their direct sum is potentially real thus corroborating the remarks made in the first paragraph of section 1. As an immediate consequence of this observation, we have (i) a direct sum representation $\mathbf{D}_{\mathrm{II}}$ consisting of only pseudo-real irreps $\mathbf{D}_i$ (for which, by definition, $\mathbf{D}_i$ and $\mathbf{D}_i^*$ are equivalent) is potentially real if the multiplicity of every irrep $\mathbf{D}_i$ appearing in $\mathbf{D}_{\mathrm{II}}$ is even, and (ii) a direct sum representation $\mathbf{D}_{\mathrm{III}}$ consisting of only essentially complex irreps is potentially real if each irrep $\mathbf{D}_i$ appearing in $\mathbf{D}_{\mathrm{III}}$ occurs such that the multiplicities of $\mathbf{D}_i$ and $\mathbf{D}_i^*$ are equal. Combining the above two results (i) and (ii) and the result that a direct sum representation $\mathbf{D}_{\mathrm{I}}$ of purely potentially real irreps is certainly potential real with theorem 1, we arrive at the following sufficient condition for the potential reality of a decomposable representation $\mathbf{D} = \mathbf{D}_{\mathrm{III}} \oplus \mathbf{D}_{\mathrm{II}} \oplus \mathbf{D}_{\mathrm{I}}$.

Theorem 2. A sufficient condition for the potential reality of a decomposable representation $\mathbf{D}$ of a group Γ is that each essentially complex irreducible component $\mathbf{D}_i$ of $\mathbf{D}$ occurs such that the multiplicities of $\mathbf{D}_i$ and $\mathbf{D}_i^*$ are equal and every pseudo-real irreducible component of $\mathbf{D}$ occurs in $\mathbf{D}$ with even multiplicity.

(6) Lastly, as in the case of irreps, we can construct any one of the three operators, namely the C-matrix $\mathbf{C}$, the bilinear metric $\mathbf{G}$ and the sesquilinear metric $\mathbf{A}$, from the other two by making use of the relation $\mathbf{C} = \alpha\mathbf{A}^{*-1}\mathbf{G}$, where α is any constant scalar (see I).

3. The reality classification of the decomposable representations of SO(3, 1)

To determine the invertible C-matrices associated with a given decomposable representation of the group SO(3, 1), we make use of theorem 1. For this purpose, it is necessary to identify the complex conjugate irrep $\mathbf{D}^*$ of a given irrep $\mathbf{D}$ of SO(3, 1). Recalling that every irrep of SO(3, 1) is characterized (Gelfand *et al* 1963) by a unique pair of invariant parameters (j_0, c), we note that the irrep $\mathbf{D}^*$ is identified once its parameter pair (j_0', c') is determined in relation to the parameter pair (j_0, c) of $\mathbf{D}$. However, since the parameters j_0' and c' of $\mathbf{D}^*$ do not change under a similarity transformation, the problem of finding j_0' and c' is solved if we determine the general conditions under which the intertwining relation

$$[\mathbf{D}(j_0, c)]^*\mathbf{K} = \mathbf{K}\mathbf{D}(j_0', c') \tag{3.1}$$

is satisfied with an invertible matrix $\mathbf{K}$. Now if $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ and $(\mathbf{A}'_\alpha, \mathbf{B}'_\alpha)$, $\alpha = 1, 2, 3$, are the infinitesimal generators (Gelfand *et al* 1963) of $\mathbf{D}(j_0, c)$ and $\mathbf{D}(j_0', c')$, respectively, then equation (3.1) would require

$$\mathbf{A}^*_\alpha\mathbf{K} = \mathbf{K}\mathbf{A}'_\alpha \qquad \mathbf{B}^*_\alpha\mathbf{K} = \mathbf{K}\mathbf{B}'_\alpha \tag{3.2}$$

where we have used the fact that the generators of $\mathbf{D}^*$ are the complex conjugates of the generators of $\mathbf{D}$. In view of the commutation relations (see Gelfand *et al* 1963) satisfied by the generators $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$, not all the six equations in (3.2) are really independent. Using the expressions given in Gelfand *et al* (1963) for the generators $(\mathbf{A}^*_\alpha, \mathbf{B}^*_\alpha)$ and $(\mathbf{A}'_\alpha, \mathbf{B}'_\alpha)$ and solving the algebraically independent relations contained in equation (3.2) (see Narahari (1986), for details), it is not difficult to see that an invertible matrix $\mathbf{K}$ exists if, and only if, $j_0 = j_0'$ and $c' = -c^*$. Thus, it follows that the complex conjugate irrep $[\mathbf{D}(j_0, c)]^*$ of an irrep $\mathbf{D}(j_0, c)$ characterized by the parameter pair (j_0, c) has parameters $(j_0, -c^*)$. Now, invoking theorem 1 we arrive at the following theorem.

Theorem 3. Let $D(j_0, c) \in NU_1 \cup NU_2$ be an essentially complex irrep of the orthochronous proper Lorentz group SO(3, 1) (see I). Then the decomposable representation $\mathbf{D}$ of SO(3, 1) containing $\mathbf{D}(j_0, c)$ admits an invertible C-matrix if, and only if, the irrep $\mathbf{D}(j_0, -c^*)$ occurs in $\mathbf{D}$ and has the same multiplicity as that of the essentially complex irrep $\mathbf{D}(j_0, c)$.

Using this theorem, we can decide whether a given decomposable representation of SO(3, 1) is essentially complex or not. In order to sort the decomposable representations possessing C-matrices further into potentially real and pseudo-real representations, we establish the following necessary and sufficient condition for potential reality of a decomposable representation of SO(3, 1).

Theorem 4. Among the decomposable representations of SO(3, 1) which admit invertible C-matrices, those, and only those, representations in which the multiplicity of every constituent pseudo-real irrep is even are potentially real (while all others are pseudo-real).

In view of theorem 2, proved in section 2, it remains only to establish that the above theorem provides a necessary condition for the potential reality of the decomposable representations of SO(3, 1). With the usual notation, let $\mathbf{D} = \mathbf{D}_{\mathrm{III}} \oplus \mathbf{D}_{\mathrm{II}} \oplus \mathbf{D}_{\mathrm{I}}$ be a potentially real decomposable representation of a group Γ. Then, by observation (2) made at the end of section 2, it follows that $\mathbf{D}$ admits at least one invertible C-matrix which satisfies $\mathbf{C}\mathbf{C}^* = \mathbf{E}$. However, since every C-matrix associated with $\mathbf{D}$ must have the structure $\mathbf{C}_{\mathrm{III}} \oplus \mathbf{C}_{\mathrm{II}} \oplus \mathbf{C}_{\mathrm{I}}$ where $\mathbf{C}_{\mathrm{III}}$, $\mathbf{C}_{\mathrm{II}}$ and $\mathbf{C}_{\mathrm{I}}$ are the invertible C-matrices admitted by the blocks $\mathbf{D}_{\mathrm{III}}$, $\mathbf{D}_{\mathrm{II}}$, and $\mathbf{D}_{\mathrm{I}}$, respectively, it is necessary that we must have $\mathbf{C}_{\mathrm{III}}\mathbf{C}^*_{\mathrm{III}} = \mathbf{E}$, $\mathbf{C}_{\mathrm{II}}\mathbf{C}^*_{\mathrm{II}} = \mathbf{E}$ and $\mathbf{C}_{\mathrm{I}}\mathbf{C}^*_{\mathrm{I}} = \mathbf{E}$, in order that $\mathbf{D}$ is potentially real. We now focus our attention on the necessary condition

$\mathbf{C}_{\mathrm{II}}\mathbf{C}^*_{\mathrm{II}} = \mathbf{E}$ and show that it implies, in turn, theorem 4 as a necessary condition for the potential reality of $\mathbf{D}$, specifically in the case of SO(3, 1). To this end, we examine the most general form of the block C-matrix $\mathbf{C}_{\mathrm{II}}$ associated with $SO(3,1)$ and check under what conditions it obeys the relation $\mathbf{C}_{\mathrm{II}}\mathbf{C}^*_{\mathrm{II}} = \mathbf{E}$.

We recall (table 1 of I) that for every pseudo-real irrep $\mathbf{D}(j_0, c)$ of SO(3, 1)

$$\mathbf{C}^2 = \mathbf{G}^2 = -\mathbf{E} \tag{3.3}$$

in the canonical Gelfand–Naimark basis. By a rearrangement of the pseudo-real irreps contained in $\mathbf{D}_{\mathrm{II}}$, we can express $\mathbf{D}_{\mathrm{II}}$ as a direct sum of the sub-blocks $\mathbf{D}^1_{\mathrm{II}}, \mathbf{D}^2_{\mathrm{II}}, \ldots, \mathbf{D}^s_{\mathrm{II}}, \ldots,$ where a typical sub-block matrix is a direct sum of equivalent pseudo-real irreps, i.e.

$$\mathbf{D}^s_{\mathrm{II}} = \mathbf{D}(j^s_0, c^s) \oplus \mathbf{D}(j^s_0, c^s) \oplus \cdots \tag{3.4}$$

with a specified multiplicity for $\mathbf{D}(j^s_0, c^s)$. However, note that the irreps belonging to different sub-blocks $\mathbf{D}^s_{\mathrm{II}}$ and $\mathbf{D}^{s'}_{\mathrm{II}}$ are not equivalent and, as a consequence, the matrix $\mathbf{C}_{\mathrm{II}}$ is also a direct sum of the sub-blocks $\mathbf{C}^1_{\mathrm{II}}, \mathbf{C}^2_{\mathrm{II}}, \ldots, \mathbf{C}^s_{\mathrm{II}}, \ldots$, which are the invertible C-matrices associated with $\mathbf{D}^1_{\mathrm{II}}, \mathbf{D}^2_{\mathrm{II}}, \ldots, \mathbf{D}^s_{\mathrm{II}}, \ldots$, respectively. Further, the condition $\mathbf{C}_{\mathrm{II}}\mathbf{C}^*_{\mathrm{II}} = \mathbf{E}$ requires that each sub-block $\mathbf{C}^s_{\mathrm{II}}$ also satisfies $\mathbf{C}^s_{\mathrm{II}}(\mathbf{C}^s_{\mathrm{II}})^* = \mathbf{E}$.

If a sub-block $\mathbf{D}^s_{\mathrm{II}}$ consists of a pseudo-real irrep of SO(3, 1) repeating an odd number of times, say k, the most general C-matrix admitted by $\mathbf{D}^s_{\mathrm{II}}$ has the block form

$$\mathbf{C}^s_{\mathrm{II}} = \begin{bmatrix} \alpha_{11}\mathbf{G} & \alpha_{12}\mathbf{G} & \alpha_{13}\mathbf{G} & \cdots \\ \alpha_{21}\mathbf{G} & \alpha_{22}\mathbf{G} & \alpha_{23}\mathbf{G} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \otimes \mathbf{G} \tag{3.5}$$

where the coefficients α_{ij} are complex numbers selected so as to make $\mathbf{C}^s_{\mathrm{II}}$ invertible. As $\mathbf{G}^2 = -\mathbf{E}$, we observe that $\mathbf{C}^s_{\mathrm{II}}(\mathbf{C}^s_{\mathrm{II}})^*$ can be the unit matrix only when complex k-dimensional matrix $[\alpha_{ij}]$ satisfies

$$[\alpha_{ij}][\alpha_{ij}]^* = -\mathbf{E}. \tag{3.6}$$

But, k is an odd number and, therefore, the k-dimensional matrix $(-\mathbf{E})$ on the right-hand side of equation (3.6) has a determinant equal to -1 while the determinant of the product matrix $[\alpha_{ij}][\alpha_{ij}]^*$ is necessarily positive. Thus, when k is an odd number, $\mathbf{C}^s_{\mathrm{II}}$ can never be chosen such that $\mathbf{C}^s_{\mathrm{II}}(\mathbf{C}^s_{\mathrm{II}})^* = +\mathbf{E}$. In other words, when $\mathbf{C}_{\mathrm{II}}\mathbf{C}^*_{\mathrm{II}} = \mathbf{E}$, none of the multiplicities k of the irreps appearing in the sub-blocks $\mathbf{C}^1_{\mathrm{II}}, \mathbf{C}^2_{\mathrm{II}}, \ldots$ can be an odd number and, hence, theorem 4 follows.

Finally, we wish to point out that the C-matrices may also be constructed from the relation $\mathbf{C} = \alpha\mathbf{A}^{*-1}\mathbf{G}$, provided we know the $\mathbf{A}$ and $\mathbf{G}$ associated with a decomposable representation. In this context, we may note that Gelfand *et al* (1963) have determined the conditions under which a decomposable representation of SO(3, 1) would preserve sesquilinear metrics and also all the sesquilinear metrics $\mathbf{A}$ associated with such a decomposable representation. In an earlier paper (Srinivasa Rao *et al* 1983), we have shown that every irrep of SO(3, 1) preserves a unique bilinear metric. Therefore, it follows at once that every decomposable representation of SO(3, 1) also preserves a bilinear metric which is a direct sum of the bilinear metrics of the constituent irreps. However, it is not difficult to see (Narahari 1986) that non-zero off-diagonal blocks can also exist in the bilinear metric whenever an irreducible component repeats itself in the given decomposable representation of SO(3, 1). In such a case, however, the scalars multiplying the off-diagonal blocks in the bilinear metric $\mathbf{G}$ have to be chosen so as to make $\mathbf{G}$ invertible.

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Appendix

We now prove theorem 1. For this purpose, it is sufficient to consider the block $\mathbf{D}_{\mathrm{III}}$ of $\mathbf{D}$. Evidently, in any block $\mathbf{D}_{\mathrm{III}}$ consisting of a direct sum of only essentially complex irreps, the multiplicity of any particular irrep $\mathbf{D}_i$ must either be equal or unequal to that of its complex conjugate $\mathbf{D}_i^*$. Therefore, we prove theorem 1 in two stages.

Case 1. The multiplicity of every irrep $\mathbf{D}_t$ occurring in $\mathbf{D}_{\mathrm{III}}$ is equal to that of its complex conjugate $\mathbf{D}_t^*$.

In this case, it is certainly possible to rearrange the order of occurrence of the irreducible components in $\mathbf{D}_{\mathrm{III}}$ such that

$$\mathbf{D}_{\mathrm{III}} = (\mathbf{D}_1 \oplus \mathbf{D}_2) \oplus (\mathbf{D}_3 \oplus \mathbf{D}_4) \oplus \cdots$$

where $\mathbf{D}_1$ is equivalent to $\mathbf{D}_2^*$, $\mathbf{D}_3$ is equivalent to $\mathbf{D}_4^*$ etc. Now, it is easy to check that $\mathbf{D}_{\mathrm{III}}$ admits the C-matrix $\mathbf{C}_{\mathrm{III}}$ given by

$$\mathbf{C}_{\mathrm{III}} = \left[\begin{array}{c|c} 0 & \mathbf{C}_{12} \\ \hline \mathbf{C}_{12} & 0 \end{array}\right] \oplus \left[\begin{array}{c|c} 0 & \mathbf{C}_{34} \\ \hline \mathbf{C}_{34} & 0 \end{array}\right] \oplus \cdots$$

where $\mathbf{C}_{12}^{-1}\mathbf{D}_2^*\mathbf{C}_{12} = \mathbf{D}_1$, $\mathbf{C}_{34}^{-1}\mathbf{D}_4^*\mathbf{C}_{34} = \mathbf{D}_3, \ldots$, and we have 'normalized' $\mathbf{C}_{12}$, $\mathbf{C}_{34}$, etc, such that $\mathbf{C}_{12}\mathbf{C}_{12}^* = \mathbf{E}$, $\mathbf{C}_{34}\mathbf{C}_{34}^* = \mathbf{E}, \ldots$, etc (cf equation (2.2) with $\alpha = 1$). Since each of the blocks $\mathbf{C}_{12}, \mathbf{C}_{34}, \ldots$ is invertible, $\mathbf{C}_{\mathrm{III}}$ is invertible. Recalling that the block C-matrices $\mathbf{C}_{\mathrm{I}}$ and $\mathbf{C}_{\mathrm{II}}$ admitted by $\mathbf{D}_{\mathrm{I}}$ and $\mathbf{D}_{\mathrm{II}}$ can always be chosen to be invertible, we conclude that $\mathbf{D} = \mathbf{D}_{\mathrm{III}} \oplus \mathbf{D}_{\mathrm{II}} \oplus \mathbf{D}_{\mathrm{I}}$ admits an invertible C-matrix $\mathbf{C} = \mathbf{C}_{\mathrm{III}} \oplus \mathbf{C}_{\mathrm{II}} \oplus \mathbf{C}_{\mathrm{I}}$. Moreover, this C-matrix evidently satisfies $\mathbf{CC}^* = \mathbf{E}$.

Case 2. $\mathbf{D}_{\mathrm{III}}$ contains at least one irreducible component, say $\mathbf{D}_t$, whose multiplicity k is not equal to the multiplicity k' of $\mathbf{D}_t^*$.

Let us rearrange the order of occurrence of the irreps contained in $\mathbf{D}_{\mathrm{III}}$ such that $\mathbf{D}_{\mathrm{III}} = \Delta_1 \oplus \Delta_2$, where Δ_1 is given by

$$\Delta_1 = (\mathbf{D}_1 \oplus \mathbf{D}_2 \oplus \cdots \oplus \mathbf{D}_k) \oplus (\mathbf{D}_{k+1} \oplus \mathbf{D}_{k+2} \oplus \cdots \oplus \mathbf{D}_{k+k'})$$

with the first k irreps all equivalent to $\mathbf{D}_t$ and the remaining k' irreps all equivalent to $\mathbf{D}_t^*$ and Δ_2 is a direct sum of the rest of the irreps contained in $\mathbf{D}_{\mathrm{III}}$. With such a rearrangement of the irreps of $\mathbf{D}_{\mathrm{III}}$, it is evident that any C-matrix $\mathbf{C}_{\mathrm{III}}$ admitted by $\mathbf{D}_{\mathrm{III}}$ also has the same structure $\mathbf{C}_{\mathrm{III}} = \mathbf{C}_1 \oplus \mathbf{C}_2$, where $\mathbf{C}_1$ is the C-matrix admitted by $\mathbf{D}_1$ and $\mathbf{C}_2$, that of $\mathbf{D}_2$. Since we are merely interested in checking whether $\mathbf{C}_{\mathrm{III}}$ is invertible or not, we can work in any basis which is convenient for this purpose. Therefore, we work in a basis in which $\Delta_1 = (\mathbf{D}_t \oplus \mathbf{D}_t \oplus \cdots \oplus \mathbf{D}_t) \oplus (\mathbf{D}_t^* \oplus \mathbf{D}_t^* \oplus \cdots \oplus \mathbf{D}_t^*)$ where (the same) $\mathbf{D}_t$ repeats k times and (the same) $\mathbf{D}_t^*$ repeats k' times. (If $\mathbf{M}_i^{-1}\mathbf{D}_i\mathbf{M}_i = \mathbf{D}_t$, $i = 1, 2, \ldots k$, and $\mathbf{M}_i^{-1}\mathbf{D}_i^*\mathbf{M}_i = \mathbf{D}_t^*$, $i = k+1, k+2, \ldots k+k'$, then $\mathbf{M}^{-1}\Delta_1\mathbf{M}$ has the above form where $\mathbf{M} = (\mathbf{M}_1 \oplus \mathbf{M}_2 \oplus \cdots \oplus \mathbf{M}_k \oplus \cdots \oplus \mathbf{M}_{k+k'})$ and $\mathbf{M}^{-1} = (\mathbf{M}_1^{-1} \oplus \mathbf{M}_2^{-1} \oplus \cdots \oplus \mathbf{M}_{k+k'}^{-1})$.) Now, writing $\mathbf{C}_1$ in block form as $[\mathbf{C}_{ij}]$, where each block $\mathbf{C}_{ij}$, $ij = 1, 2, \ldots k+k'$, is a 'square matrix' having the 'dimension' of $\mathbf{D}_t$ ($\mathbf{D}_t$ and $\mathbf{D}_t^*$ are evidently of the same dimension), and using it in the relation $\mathbf{C}_1\Delta_1 = \Delta_1^*\mathbf{C}_1$, we note that the blocks $\mathbf{C}_{ij}$ vanish when $i, j = 1, 2, \ldots, k$, or $i, j = k+1, k+2, \ldots, k+k'$, because these block C-matrices intertwine $\mathbf{D}_t$ with $\mathbf{D}_t^*$ and, hence, must necessarily vanish as $\mathbf{D}_t$ is essentially complex. Further, since the blocks $\mathbf{C}_{ij}$, when $i = 1, 2, \ldots, k$, $j = k+1, k+2, \ldots, k+k'$, or $i = k+1, k+2, \ldots, k+k'$, $j = 1, 2, \ldots k$, intertwine $\mathbf{D}_t^*$ and $\mathbf{D}_t$ with themselves, respectively, it follows, in view of the irreducibility of $\mathbf{D}_t$ and the Schur lemma, that each such block $\mathbf{C}_{ij}$ is a scalar multiple of the unit matrix $\mathbf{E}$ (having the dimension of $\mathbf{D}_t$). Therefore, $\mathbf{C}_1$ has the block-structure

$$\mathbf{C}_1 = \left[\begin{array}{c|c} 0 & \mathbf{P} \\ \hline \mathbf{Q} & 0 \end{array}\right]$$

with the 'rectangular blocks' $\mathbf{P}$ and $\mathbf{Q}$ given by $\mathbf{P} = [p_{ij}\mathbf{E}]$, $i = 1, 2, \ldots, k$, $j = k+1, k+2, \ldots, k+k'$ and $\mathbf{Q} = [q_{ij}\mathbf{E}]$, $i = k+1, k+2, \ldots, k+k'$, $j = 1, 2, \ldots, k$, where p_{ij} and q_{ij} are arbitrary complex numbers. Let us introduce the $(k+k')$-dimensional (square) matrix

$$\mathbf{Z} = [z_{ij}] = \left[\begin{array}{c|c} 0 & [p_{ij}] \\ \hline [q_{ij}] & 0 \end{array}\right]$$

where

$$z_{ij} = \begin{cases} 0 & \text{if } i, j = 1, 2, \ldots, k \qquad \text{or } i, j = k+1, k+2, \ldots, k+k' \\ p_{ij} & \text{if } i = 1, 2, \ldots, k \qquad j = k+1, k+2, \ldots, k+k' \\ q_{ij} & \text{if } i = k+1, k+2, \ldots, k+k' \qquad j = 1, 2, \ldots, k. \end{cases}$$

Then, $\mathbf{C}_1$ may be expressed as the direct product $\mathbf{C}_1 = \mathbf{Z} \otimes \mathbf{E}$. Further, we note that with a real permutation matrix $\mathbf{S}$ defined by

$$S(i, j; k, l) = \delta_{jk}\delta_{il} \qquad \text{and} \qquad \mathbf{S} = \mathbf{S}^{-1}$$

where $S(i, j; k, l)$ is the element of the matrix $\mathbf{S}$ occurring in the (ij)th row and (kl)th column, $\mathbf{C}_1$ is transformed into $\mathbf{E} \otimes \mathbf{Z}$, i.e.

$$\mathbf{C}_1 \mapsto \mathbf{C}_1' = \mathbf{S}^{*-1}\mathbf{C}_1\mathbf{S} = \mathbf{E} \otimes \mathbf{Z}.$$

Now, a direct evaluation of $\det(\mathbf{Z})$ by the Laplace method shows that $\det(\mathbf{Z}) = 0$ so that $\mathbf{Z}$ is singular. Hence, $\mathbf{C}_1' = \mathbf{E} \otimes \mathbf{Z} = \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots$ is not invertible and, hence, $\mathbf{C}_1$ is also not invertible. Thus, $\mathbf{D}_{\mathrm{III}}$, and hence $\mathbf{D}$, does not possess invertible C-matrices.

Theorem 1 follows by collecting together the conclusions arrived at in cases 1 and 2.

References

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